

Painlevé III and 2D Polymers

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Abstract

Recently the scaling function of the dilute non-contractible self-avoiding 2D polymer loop on a cylinder was related to the Painlevé III transcendent. Using the perturbation theory, the thermodynamic Bethe ansatz and numerical calculations we argue a similar relation for the contractible self-avoiding loop.

1. Introduction

Among the recent studies in the 2D integrable relativistic field theory (RFT) one of the most impressive results was achieved by P.Fendley and H.Saleur in ref.[1] (see also related ref.[2]). The authors have succeeded to link the universal scaling function of a single 2D self-avoiding loop winding once around a cylinder to a particular Painlevé III transcendent. After almost 20 years of the famous two-spin Ising scaling correlation function [3] standing alone, a non-linear differential equation appears again in 2D RFT to govern the universal scaling behavior. Being related to an apparently interacting field theory the result of [1] seems extremely suggestive. Moreover, it solves exactly an important problem in 2D polymer statistics and is directly comparable with series expansions, simulations etc. It should be mentioned that the discovery was preceded by a series of fascinating studies in the topological and $N = 2$ supersymmetric 2D RFT's [4–6]. Although it is not obvious that it is the $N = 2$ SUSY that plays the most important role in the phenomenon of [1], the physical significance of ref.[4] worth to be better understood (see ref.[2] in this connection).

The problem of 2D self-avoiding polymers is formulated basically as follows. Consider some 2D lattice (say the honeycomb one to get rid of any problems with self-avoiding) and count the continuous self-avoiding paths (closed or open) through the links of the lattice. In general there are several disconnected components, each of them being called the polymer (respectively closed and open). Because of the self-avoiding there are many topologically

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different configurations and therefore an innumerable amount of various counting problems. As a first step one tries to count the closed polymer configurations taking into account the number of separate polymers. The configurations can be generated e.g. by the K -expansion of the $O(n)$ -model partition function

$$Z(K, n) = \text{tr} \prod_{\langle i, j \rangle} (1 + K s_i^a s_j^a) \quad (1.1)$$

where the product is over the nearest neighbour pairs of sites $\langle i, j \rangle$. The n -component spins s_i^a ; $a = 1, 2, \dots, n$ sit at the sites and are chosen so that $\text{tr} s^a = 0$, $\text{tr} s^a s^b = \delta^{ab}$ and $\text{tr} s^a s^b s^c = 0$. For (1.1) we have

$$Z(K, n) = \sum_{\substack{\text{loop} \\ \text{configurations}}} K^{\# \text{ of links}} n^{\# \text{ of loops}} \quad (1.2)$$

As usual for a homogeneous lattice of large volume V the thermodynamic limit is supposed to exist

$$\mathcal{F}(K, n) = -\frac{1}{V} \lim_{V \rightarrow \infty} \log Z(K, n) \quad (1.3)$$

defining the specific free energy $\mathcal{F}(K, n)$.

What is interesting for the field theory is the statistics of large loop configurations (i.e. with the number of lattice links in a polymer tending to infinity) which is believed to be universal, i.e. independent on the lattice realization. The loops blow up near some critical value K_c where the statistical quantities (like the free energy) develop singularities characteristic for the second order phase transition. The phase at $K < K_c$ is called the dilute polymer one. While $K < K_c$ the correlation length (i.e. the typical spatial extension of the polymers), which we denote M^{-1} here, remains finite but diverges as

$$M^{-1} \sim (K_c - K)^{-(p+1)/4} \quad (1.4)$$

The standard parameter p which appears in the exponent of eq.(1.4) is related to n as follows

$$n = 2 \cos \frac{\pi}{p} \quad (1.5)$$

Near the critical point the observables (correlation functions etc.) apporportion some singular in $K_c - K$ parts, which depend only on the distances and extents scaled in the units of M^{-1} and bear the rotational (Lorentz) symmetry characteristic for RFT. Being universal these scaling functions are objects of the field theory. E.g. the free energy (1.3) itself behaves as

$$\mathcal{F}(K, n) = \tilde{\mathcal{F}}(K, n) + M^2 f(n) \quad (1.6)$$

where $\tilde{\mathcal{F}}$ is some non-universal background regular in $K - K_c$ which is typically of no interest for the field theory. Contrary, the second piece in (1.6) is contributed by the asymptotically large polymer loops and describes their universal statistics independent on

how the polymers are arranged on the lattice level. For the 2D polymer problem $f(n)$ is known exactly

$$f(n) = -\frac{1}{4} \tan \frac{\pi p}{2} \quad (1.7)$$

This result comes both from the relation to the sin-Gordon model (see below) where the exact vacuum energy was found by different methods [7–9] and from a formal analytic continuation of the thermodynamic Bethe ansatz (TBA) equations at integer p [10].

From eq.(1.2) it is clear that the n -expansion

$$f(n) = \sum_k f_k n^k = -\frac{1}{4}n - \frac{1}{4\pi}n^2 - \frac{1}{4\pi^2} \left(1 + \frac{3\pi^2}{8}\right) n^3 + \dots \quad (1.8)$$

is in fact the expansion in the number of separate polymer loops. E.g. the first number $f_1 = -1/4$ corresponds to the (scaling) internal statistical weight (or activity) of a single isolated loop. In other words f_1 is the specific free energy of a dilute gas of non-interacting contractible polymer loops. The second two-polymer number $f_2 = -1/4\pi$ receives in fact contributions from two topologically different polymer configurations drawn in fig.1. The first one (fig.1a) gives the first virial coefficient for the gas of contractible loops which takes into account their interaction (due to the self-avoiding) at the two-component cluster level. The second contribution (fig.1b) is the activity of the structure with two nested loops. Further numbers in (1.8) include more and more distinct topological contributions. It is an interesting problem (unsolved to my knowledge) to separate them.

However, in the present note we deal with another settlement of the polymer problem. In the next section the counting problem of closed polymers on an infinite cylinder is considered and the corresponding scaling functions are defined. Then the remarkable result of ref.[1] is quoted which relates explicitly the one non-contractible polymer scaling function to a particular Painlevé III transcendent. We present also a similar relation for the scaling function of a single contractible polymer. Sect.3 contains few details and propositions about the Painlevé III transcendents including their representations as Fredholm determinants as well as the TBA-like representations [1,2]. In sect.4 we discuss the useful relation between the cylinder polymer scaling functions and the finite-temperature free energy of the sin-Gordon model (SG) in the repulsive regime $\beta^2 > 4\pi$. In particular this includes the restricted sin-Gordon models (RSG), i.e., the Φ_{13} perturbed minimal conformal field theories (CFT's). Few first terms in the perturbative expansions of the SG and RSG finite-temperature free energy are developed in sect.5. They are to be compared with the corresponding expansion of the Painlevé III. Sect.6 contains the TBA considerations in the sin-Gordon model near its $N = 2$ supersymmetric point $\beta^2 = 16\pi/3$. These permit us to derive the TBA-like formulas quoted in sect.3. A bulk of questions and hints arises in connection with the things considered. Some of them are listed in sect.7.

2. Polymers on a cylinder. The Painlevé III

Let us start again from the lattice level. Imagine a long cylinder made of say again the honeycomb lattice (fig.2) and denote L the length of the cylinder and R its circum-

ference. When counting the closed polymer configurations on this cylinder lattice one can distinguish between two kinds of polymers, the “winding loops” which wind once around the cylinder (fig.3a) (it is plain that the self-avoiding allows a closed polymer to do it only once) and the “non-winding loops” which don’t (fig.3b). We shall count these two kinds separately defining the partition function as

$$Z(m, n|K, R, L) = \sum_{\text{cylinder polymer configurations}} K^{\# \text{ of links}} m^{\# \text{ of winding loops}} n^{\# \text{ of non-winding loops}} \quad (2.1)$$

where m is the weight of a winding polymer and n is that of a non-winding one. At $L \rightarrow \infty$ as usual we introduce the free energy per unit length

$$\mathcal{E}(m, n|K, R) = -\frac{1}{L} \lim_{L \rightarrow \infty} \log Z(m, n|K, R, L) \quad (2.2)$$

Of course, at R fixed (in the lattice units) $\mathcal{E}(m, n|K, R)$ shows no criticality near K_c . It does however if R simultaneously goes to infinity as fast as the correlation length M^{-1} does. Defining the scaling circumference

$$t = MR \quad (2.3)$$

we expect that at t fixed and $K \rightarrow K_c - 0$

$$\mathcal{E}(m, n|K, R) = \tilde{\mathcal{E}}(m, n|K, R) + MF(m, n|t) \quad (2.4)$$

Here $\tilde{\mathcal{E}}(m, n|K, R)$ is non-singular at K_c and in fact for R large enough

$$\tilde{\mathcal{E}}(m, n|K, R) = R\tilde{\mathcal{F}}(K, n) \quad (2.5)$$

independently on m (with the same $\tilde{\mathcal{F}}(K, n)$ as in eq.(1.6)). What is interesting for us is the universal scaling function $F(m, n|t)$ which describes the statistics of large (dilute) polymers on the cylinder. Obviously at $t \rightarrow \infty$ ($R \gg M^{-1}$) the influence of the cylinder geometry disappears and

$$F(m, n|t) \sim tf(n), \quad t \rightarrow \infty \quad (2.6)$$

with $f(n)$ defined in the previous section. For what follows it is convenient to introduce also the special notations

$$F_+(n|t) = F(n, n|t) \quad (2.7)$$

for the scaling function of equally weighted winding and non-winding loops and also

$$F_-(n|t) = F(-n, n|t) \quad (2.8)$$

for that weighted with the opposite signs.

Again the double expansion in m and n

$$\begin{aligned} F(m, n|t) &= \sum_{l_1, l_2} F_{l_1, l_2}(t) m^{l_1} n^{l_2} \\ &= mF_{1,0}(t) + nF_{0,1}(t) + m^2F_{2,0}(t) + mnF_{1,1}(t) + n^2F_{0,2}(t) + \dots \end{aligned} \quad (2.9)$$

isolates the configurations with fixed numbers l_1 of winding loops and l_2 of non-winding ones. Each of the linear terms $F_{1,0}(t)$ and $F_{0,1}(t)$ corresponds respectively to the scaling statistical weight (per unit length of the cylinder) of a single isolated non-contractible (fig.3a) and contractible (fig.3b) polymer loop. The next quadratic terms in expansion (2.9) contain different two-loop scaling functions. Namely, $F_{2,0}(t)$ and $F_{1,1}(t)$ are related to the two-polymer clusters drawn in figs.4a and 4b while $F_{0,2}(t)$ is contributed by two topologically different configurations of figs.4c and 4d.

The one-loop cylinder scaling functions are of primary interest below so that we introduce special notations for them

$$\begin{aligned} F_n(t) &= F_{1,0}(t) \\ F_c(t) &= F_{0,1}(t) \end{aligned} \quad (2.10)$$

In connection with the functions (2.7) and (2.8) we shall also consider the combinations

$$F_{\pm}(t) = F_c(t) \pm F_n(t) \quad (2.11)$$

The remarkable result of ref.[1] is that

$$F_n(t) = \frac{dU(t)}{dt} \quad (2.12)$$

where $U(t)$ is a particular solution (regular at $t > 0$) to the Painlevé III equation

$$\frac{1}{t} \frac{d}{dt} t \frac{d}{dt} U = \frac{1}{2} \sinh 2U \quad (2.13)$$

This equation admits a one parameter family of regular at $t > 0$ solutions (see e.g.[11]) called the Painlevé III transcendents. The special solution in eq.(2.12) is fixed by the following boundary conditions at $t \rightarrow 0$

$$U(t) = -\frac{1}{3} \log t - \frac{1}{2} \log \frac{\kappa}{4} + O(t^{4/3}) \quad (2.14)$$

where we have denoted

$$\kappa = \frac{\Gamma^2(1/3)}{\Gamma^2(2/3)} = 3.91392514\dots \quad (2.15)$$

From eqs.(2.13) and (2.14) it is quite easy to see that $U(t)$ is essentially a regular series

$$U(t) = -\frac{1}{4} \log \xi - \frac{1}{2} \log \frac{\kappa}{4} + \sum_{n=1}^{\infty} U_n \xi^n \quad (2.16)$$

in the variable

$$\xi = t^{4/3} \quad (2.17)$$

The coefficients U_n can be recursively generated by the equation (2.13)

$$\frac{16}{9} \sum_{n=1}^{\infty} n^2 U_n \xi^n = \frac{\xi}{\kappa} e^{2 \sum_{n=1}^{\infty} U_n \xi^n} - \frac{\xi^2 \kappa}{16} e^{-2 \sum_{n=1}^{\infty} U_n \xi^n} \quad (2.18)$$

This way one finds

$$\begin{aligned} tF_n(t) &= -\frac{1}{3} + \frac{4}{3} \sum_{n=1}^{\infty} n U_n \xi^n \\ &= -\frac{1}{3} + \frac{3}{4\kappa} \xi - \frac{3(\kappa^3 - 18)}{128\kappa^2} \xi^2 + \frac{27(\kappa^3 + 18)}{2048\kappa^3} \xi^3 - \frac{27(\kappa^6 + 30\kappa^3 - 648)}{131072\kappa^4} \xi^4 + \dots \end{aligned} \quad (2.19)$$

On the other hand the Painlevé III transcendent $U(t)$ admits the following infinite series representation [11,2]

$$U(t) = \sum_{k=0}^{\infty} \frac{2}{2k+1} \int \prod_{i=1}^{2k+1} \frac{e^{-t \cosh \theta_i}}{\cosh \frac{\theta_i - \theta_{i+1}}{2}} \frac{d\theta_i}{4\pi} \quad (2.20)$$

where it is implied that $\theta_{2k+2} = \theta_1$. The series is convergent for all $t > 0$ while the first terms control the $t \rightarrow \infty$ asymptotics. In particular the scaling function $F_n(t)$ falls off exponentially at $t \rightarrow \infty$

$$F_n(t) = -\frac{1}{\pi} K_1(t) + O(e^{-2t}) \quad (2.21)$$

where

$$K_\nu(t) = \frac{1}{2} \int e^{\nu\theta - t \cosh \theta} d\theta \quad (2.22)$$

is the modified Bessel function. The non-contractible one-loop function $F_n(t)$ is plotted in fig.5.

Below we argue that the contractible one-loop scaling function $F_c(t)$ satisfies the following equation

$$\frac{1}{t} \frac{d}{dt} t F_c(t) = -\frac{1}{2} \cosh 2U \quad (2.23)$$

where $U(t)$ is the same Painlevé III transcendent (2.20) as for the non-contractible function $F_n(t)$.

Since $U(t) \rightarrow 0$ at $t \rightarrow \infty$ it follows from eq.(2.23) that $F_c(t)$ has a linear leading asymptotic in this limit

$$F_c(t) \sim -\frac{t}{4} \quad \text{at } t \rightarrow \infty \quad (2.24)$$

in agreement with (2.6) and the expansion (1.8) of the bulk free energy. Eq.(2.23) defines $F_c(t)$ up to an item of C/t with an arbitrary constant C . It seems natural to choose C so that the corrections to (2.24) at $t \rightarrow \infty$ would be exponentially small. Then we have

$$F_c(t) = -\frac{t}{4} + \frac{1}{\pi^2 t} \int_t^\infty K_0^2(r) r dr + O(e^{-4t}) \quad (2.25)$$

In the next section it is argued that in fact $F_c(t)$ develops a series representation quite similar to eqs.(2.12) and (2.20). Define $V(t)$ as

$$V(t) = \frac{t^2}{8} + \sum_{k=1}^\infty \frac{2}{2k} \int \prod_{i=1}^{2k} \frac{e^{-t \cosh \theta_i}}{\cosh \frac{\theta_i - \theta_{i+1}}{2}} \frac{d\theta_i}{4\pi} \quad (2.26)$$

where again $\theta_{2k+1} = \theta_1$ in each term in the r.h.s. (note that now we sum up the “even loops” instead of the “odd” ones in eq.(2.20)). Then

$$F_c(t) = -\frac{dV(t)}{dt} \quad (2.27)$$

At small t the contractible loop scaling function is again a regular series in the variable (2.17)

$$tF_c(t) = c - \frac{4}{3} \sum_{n=1}^\infty n V_n \xi^n \quad (2.28)$$

The coefficients V_n are readily restored from eq.(2.23)

$$\frac{16}{9} \sum_{n=1}^\infty n^2 V_n \xi^n = \frac{\xi}{\kappa} e^{2 \sum_{n=1}^\infty U_n \xi^n} + \frac{\xi^2 \kappa}{16} e^{-2 \sum_{n=1}^\infty U_n \xi^n} \quad (2.29)$$

As it was mentioned before, the constant term c in (2.28) is not fixed by eq.(2.23). We shall see shortly that the above choice of the integration constant C in eq.(2.25) corresponds to $c = 1/18$. Expansion (2.28) reads

$$tF_c(t) = \frac{1}{18} - \frac{3}{4\kappa} \xi - \frac{3(\kappa^3 + 18)}{128\kappa^2} \xi^2 + \frac{9(5\kappa^3 - 54)}{2048\kappa^3} \xi^3 - \frac{27(\kappa^6 + 6\kappa^3 + 648)}{131072\kappa^4} \xi^4 + \dots \quad (2.30)$$

The contractible scaling function $F_c(t)$ is presented in fig.6 (without the “bulk energy” term $-t/4$).

For the later use we quote also the short-distance expansions of the combinations (2.11)

$$tF_+(t) = -\frac{5}{18} - \frac{3\kappa}{64} \xi^2 + \frac{9}{256} \xi^3 - \frac{27(\kappa^3 + 18)}{65536\kappa} \xi^4 + \dots \quad (2.31)$$

and

$$tF_-(t) = \frac{7}{18} - \frac{3}{2\kappa} \xi - \frac{27}{32\kappa^2} \xi^2 + \frac{9(\kappa^3 - 54)}{1024\kappa^3} \xi^3 + \frac{81(\kappa^3 - 54)}{16384\kappa^4} \xi^4 + \dots \quad (2.32)$$

These two functions are plotted in fig.7. They satisfy the equations

$$\frac{1}{t} \frac{d}{dt} t F_{\pm}(t) = -\frac{1}{2} e^{\mp 2U} \quad (2.33)$$

3. Painlevé III vs. Fredholm theory

We begin with the second kind Fredholm equation

$$f(\theta) - \lambda \int_{-\infty}^{\infty} K(\theta, \theta') f(\theta') d\theta' = g(\theta) \quad (3.1)$$

where $K(\theta, \theta') = K(\theta', \theta)$ is a symmetric kernel (bounded in $L_2(-\infty, \infty)$) and λ is the spectral parameter. Below we take the kernel of the following special form

$$K(\theta, \theta') = \frac{e^{-u(\theta)-u(\theta')}}{\cosh \frac{\theta - \theta'}{2}} \quad (3.2)$$

with some suitably chosen “potential” $u(\theta)$. For the Painlevé III theory $2u(\theta) = t \cosh \theta$ will be relevant. Note, that in terms of $x = e^{\theta}$ and substituting $f(\theta) = \exp(\theta/2) \tilde{f}(x)$, $g(\theta) = \exp(\theta/2) \tilde{g}(x)$ eq.(3.1) reads

$$\tilde{f}(x) - 2\lambda \int_0^{\infty} \frac{e^{-u(x)-u(x')}}{x+x'} \tilde{f}(x') dx' = \tilde{g}(x) \quad (3.3)$$

The spectral information about (3.1) is contained in the Fredholm determinant $D(u|\lambda)$, which is entire function of λ with zeroes located at the eigenvalues λ_a , $a = 1, 2, \dots$ of (3.1). It can be calculated as a convergent series in λ

$$D(u|\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} D_n(u) \quad (3.4)$$

with

$$\begin{aligned} D_n(u) &= \int_0^{\infty} \det \left(\frac{1}{x_i + x_j} \right) \prod_{i=1}^n 2e^{-2u(x_i)} dx_i \\ &= \int_0^{\infty} \prod_{i>j}^n \frac{(x_i - x_j)^2}{(x_i + x_j)^2} \prod_{i=1}^n e^{-2u(x_i)} \frac{dx_i}{x_i} \end{aligned} \quad (3.5)$$

Denote

$$-W(u|\lambda) = \log D(u|\lambda) \quad (3.6)$$

For $W(u|\lambda)$ we have the following λ -series

$$W(u|\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n} W_n(u) \quad (3.7)$$

convergent while $|\lambda^{-1}| > \sup_a |\lambda_a^{-1}|$. Here

$$W_n(u) = \int \prod_{i=1}^n \frac{e^{-2u(\theta_i)}}{\cosh \frac{\theta_i - \theta_{i+1}}{2}} d\theta_i = \int_0^\infty \prod_{i=1}^n \frac{2e^{-2u(x_i)}}{x_i + x_{i+1}} dx_i \quad (3.8)$$

where we imply that $\theta_{n+1} = \theta_1$ or $x_{n+1} = x_1$.

A link to integrable non-linear equations appears if we take

$$u(\theta) = \frac{t}{4}e^\theta + \frac{\bar{t}}{4}e^{-\theta} + r(\theta) \quad (3.9)$$

with some $r(\theta)$ and denote

$$U(u|\lambda) = W(u|\lambda) - W(u|-\lambda) = \log \frac{D(u|\lambda)}{D(u|-\lambda)} \quad (3.10)$$

Then it happens that (we denote $\partial = \partial/\partial t$ and $\bar{\partial} = \partial/\partial \bar{t}$)

$$4\partial\bar{\partial}W(u|\lambda) = \frac{1}{4} \left(e^{2U(u|\lambda)} - 1 \right) \quad (3.11)$$

or in terms of the determinants $D(u|\lambda)$

$$D(u|\lambda)\partial\bar{\partial}D(u|\lambda) - \partial D(u|\lambda)\bar{\partial}D(u|\lambda) = \frac{1}{16} (D^2(u|\lambda) - D^2(u|-\lambda)) \quad (3.12)$$

In particular $U = U(u|\lambda) = -U(u|-\lambda)$ solves the 2D sinh-Gordon equation

$$4\partial\bar{\partial}U = \frac{1}{2} \sinh 2U \quad (3.13)$$

Eq.(3.11) can be verified comparing order by order the λ -expansions in the right and left hand sides. This would require the following relations between the integrals (3.8)

$$\frac{16}{n} \partial\bar{\partial}W_n(u) = \sum_{\left\{ \substack{p_1, p_3, p_5, \dots \geq 0 \\ p_1 + 3p_3 + 5p_5 + \dots = n} \right\}} \prod_{k=0}^{\infty} \frac{1}{p_{2k+1}!} \left(\frac{4W_{2k+1}(u)}{2k+1} \right)^{p_{2k+1}} \quad (3.14)$$

E.g.,

$$\begin{aligned} 4\partial\bar{\partial}W_1 &= W_1 \\ 4\partial\bar{\partial}W_2 &= 4W_1^2 \\ 4\partial\bar{\partial}W_3 &= 8W_1^3 + W_3 \\ 4\partial\bar{\partial}W_4 &= \frac{32}{3}W_1^4 + \frac{16}{3}W_1W_3 \end{aligned} \quad (3.15)$$

etc. The first three relations in (3.15) are verified directly by comparing the corresponding integrands. Starting from $n = 4$ one has to perform a symmetrisation in the rational in x_i , $i = 1, 2, \dots, n$ part of the integrands (this is allowed due to the permutation symmetry of the rest part $\prod_{i=1}^n \exp(-2u(x_i))dx_i$, common for the both sides of eq.(3.14)). I have checked (3.14) up to $n = 6$. The complete proof perhaps can be found following the lines of ref.[11]. It should be stressed that the validity of (3.14) is of purely combinatorial nature, being based on certain algebraic identities between the symmetrized rational expressions in x_i . Therefore we expect eqs.(3.11–14) to hold independently on λ and even on the choice of the “residual” potential $r(\theta)$ in eq.(3.9). Moreover, they remain valid if in eq.(3.1) (and therefore also in eq.(3.8)) we replace the whole real axis by any other sensible integration contour in θ . In this note we shall not develop further these lines of generalizations.

Before turning at the rotationally symmetric (isotropic) version of eq.(3.11) it is worth to mention the following interesting observation of refs.[1,2]. Consider a small variation $\delta u(\theta)$ of the potential in (3.2). Then

$$\delta W(u|\lambda) = -2\lambda \int R(\theta|\lambda) \delta u(\theta) d\theta \quad (3.16)$$

where $R(\theta|\lambda) = \mathcal{R}(\theta, \theta|\lambda)$ and $\mathcal{R}(\theta, \theta'|\lambda) = \mathcal{R}(\theta', \theta|\lambda)$ is the resolvent kernel of eq.(3.1), i.e., the (unique at $\lambda \neq \lambda_a$) solution to

$$\mathcal{R}(\theta, \theta'|\lambda) - \lambda \int K(\theta, \theta'') \mathcal{R}(\theta'', \theta'|\lambda) d\theta'' = K(\theta, \theta') \quad (3.17)$$

As a series in λ it reads

$$R(\theta|\lambda) = \sum_{n=0}^{\infty} \lambda^n R_n(\theta) \quad (3.18)$$

where

$$R_n(\theta) = e^{-2u(\theta)} \int \frac{e^{-2u(\theta_1) - \dots - 2u(\theta_n)}}{\cosh \frac{\theta - \theta_1}{2} \cosh \frac{\theta_1 - \theta_2}{2} \dots \cosh \frac{\theta_n - \theta}{2}} d\theta_1 \dots d\theta_n \quad (3.19)$$

We define also

$$\begin{aligned} 2R_+(\theta|\lambda) &= R(\theta|\lambda) + R(\theta|-\lambda) = 2 \sum_{k=0}^{\infty} \lambda^{2k} R_{2k}(\theta) \\ 2R_-(\theta|\lambda) &= R(\theta|\lambda) - R(\theta|-\lambda) = 2 \sum_{k=0}^{\infty} \lambda^{2k+1} R_{2k+1}(\theta) \end{aligned} \quad (3.20)$$

In refs.[1,2] on the basis of the field theory considerations quantity $R_+(\theta|\lambda)$ was related to the following “TBA-like” system of non-linear integral equations

$$2u(\theta) = \varepsilon(\theta) + \int_{-\infty}^{\infty} \frac{\log(1 + \eta^2(\theta'))}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi} \quad (3.21a)$$

$$\eta(\theta) = 2\lambda \int \frac{e^{-\varepsilon(\theta')}}{\cosh(\theta - \theta')} d\theta' \quad (3.21b)$$

with the same potential $u(\theta)$ and the same λ . It was argued that there is a solution $\varepsilon(\theta)$, $\eta(\theta)$ to (3.21) such that

$$R_+(\theta|\lambda) = e^{-\varepsilon(\theta)} \quad (3.22)$$

In sect.6 of this note we extend slightly the TBA considerations of [1,2] to argue a similar formula for $R_-(\theta|\lambda)$

$$R_-(\theta|\lambda) = R_+(\theta|\lambda) \int_{-\infty}^{\infty} \frac{\arctan \eta(\theta')}{\cosh^2(\theta - \theta')} \frac{d\theta'}{\pi} \quad (3.23)$$

Strictly speaking, the arguments of [1,2] (and also of sect.6 below) are relevant only for the isotropic case $2u(\theta) = t \cosh \theta$. However, if we start to solve the system (3.21) as a power series in λ we would encounter for the corresponding terms in (3.22) and (3.23) expressions like

$$\int e^{-2u(\theta_1) - \dots - 2u(\theta_n)} I(\theta_1, \dots, \theta_n) d\theta_1 \dots d\theta_n \quad (3.24)$$

Here the “potential weighted” integrations over $\theta_1, \dots, \theta_n$ arise from the iterations of eq.(3.21b) while $I(\theta_1, \dots, \theta_n)$ results from the “intermediate” integrations in the first equation (3.21a) or in eq.(3.23). They turn to be rational functions of $x_i = \exp \theta_i$; $i = 1, 2, \dots, n$ which, after symmetrization, are expected to reproduce the (symmetrized) rational parts in the integrands of (3.19). In first few orders this can be verified explicitly. If (3.21–23) is true in the isotropic case (as it was argued in [1,2] and sect.6) this phenomenon must persist at all orders, since it is quite hard to imagine that the transcendental integrations over $\theta_1, \dots, \theta_n$ may play any role in relating (3.24) to (3.19). One concludes that the TBA representation (3.21–23) is of more general nature and must hold for any reasonable potential $u(\theta)$. Also, corresponding to different possible choices of the integration contour in (3.1) we have to change the contour in (3.21b). Since the θ -integrations in eqs.(3.21a) and (3.23) are important in the building of $I(\theta_1, \dots, \theta_n)$, the contours there must remain unchanged. Of course, the consideration above is not a proof, which is lacking up to now. I have verified the TBA representation numerically for several (presumably random) samples of $u(\theta)$.

The TBA representation (although unproven) turns extremely useful in the numerical calculations. E.g., for the isotropic case (3.25) with $|\lambda| < 2\pi$ (and not too close to 2π) the iterative solution of the system (3.21–23) converges fast and works much better than both the Painlevé equation and the direct calculation of the Fredholm determinant. This typically persists for other samples of $u(\theta)$ with $|\lambda^{-1}|$ well below the $\sup_a |\lambda_a^{-1}|$. In particular, the curves of figs.5–7 are computed in very this way.

From now on we switch to the rotationally symmetric (i.e., invariant under $t \rightarrow \Lambda t$, $\bar{t} \rightarrow \Lambda^{-1} \bar{t}$) solutions to (3.11–13). This corresponds to taking in (3.9) $r(\theta) = \text{const}$ (which can be absorbed by the spectral parameter) and keeping $(-\infty, \infty)$ as the integration region in (3.1). Thus we restrict to $t = \bar{t}$ real and

$$2u(\theta) = t \cosh \theta \quad (3.25)$$

In eqs.(3.11–13) we can substitute $2\partial = 2\bar{\partial} = d/dt$, $4\partial\bar{\partial} = d^2/dt^2 + t^{-1}d/dt$ so that (3.13) is reduced to the Painlevé III equation (2.13). The series (3.7) turns also the large t expansion and we have

$$W(t|\lambda) = 2\lambda K_0(t) + 8\lambda^2 \left[-\log t \int_t^\infty K_0^2(r) r dr + \int_t^\infty K_0^2(r) r \log r dr \right] + O(\lambda^3 e^{-3t}) \quad (3.26)$$

Considering real solutions we have to take λ real. It is possible to show that at real $t > 0$ $\sup_a \lambda_a^{-1} < 2\pi$. Therefore $W(t|\lambda)$ is regular at $t > 0$ for $\lambda \leq (2\pi)^{-1}$. At $2\pi\lambda > 1$ the Fredholm determinant $D(t|\lambda)$ has infinite number of zeroes on the positive t axis, $t = 0$ being their condensation point [11]. We shall consider the case $2\pi|\lambda| \leq 1$ where all the functions appearing in eqs.(3.6–13) are regular at $t > 0$. Denote

$$2\pi\lambda = \sin \frac{\pi\sigma}{2} \quad (3.27)$$

with real $-1 \leq \sigma \leq 1$. It is relatively easy to analyse the $t \rightarrow 0$ behavior of the determinant (3.4) with the potential (3.25). One finds

$$W(t|\lambda) = \frac{\sigma(\sigma+2)}{4} \log \frac{8}{t} + B(\sigma) + O(t^{2\pm 2\sigma}) \quad (3.28)$$

where $(\psi(x) = \Gamma'(x)/\Gamma(x))$

$$B(\sigma) = \frac{1}{4} \int_0^\sigma (1+x) \left[\psi\left(\frac{1+x}{2}\right) + \psi\left(\frac{-1-x}{2}\right) - 2 \right] dx \quad (3.29)$$

In particular, for the combination (3.10) we have

$$U(t|\lambda) = \sigma \log \frac{8}{t} - \frac{1}{2} \log \kappa(\sigma) + O(t^{2\pm 2\sigma}) \quad (3.30)$$

with

$$\kappa(\sigma) = \exp(2B(-\sigma) - 2B(\sigma)) = \frac{\Gamma^2\left(\frac{1}{2} - \frac{\sigma}{2}\right)}{\Gamma^2\left(\frac{1}{2} + \frac{\sigma}{2}\right)} \quad (3.31)$$

Using (3.30) and eqs.(3.11–13) one can systematically recover the further $t \rightarrow 0$ corrections to (3.28). They appear as

$$W(t|\lambda) = \frac{(\sigma+1)^2 - 1}{4} \log \frac{8}{t} - \frac{t^2}{16} + \sum_{m,n=0}^\infty B_{m,n}(\sigma) \left(\frac{t}{8}\right)^{2\alpha_{m,n}} \quad (3.32)$$

where $\alpha_{m,n} = (m+n) + \sigma(m-n)$. In this double series

$$B_{0,0}(\sigma) = B(\sigma) \quad (3.33)$$

and the next coefficients $B_{m,n}(\sigma)$ are found order by order from the following relation

$$\frac{1}{4} \sum_{m,n=0}^{\infty} \alpha_{m,n}^2 B_{m,n}(\sigma) x^{2\alpha_{m,n}} = \frac{x^{2\alpha_{0,1}}}{\kappa(\sigma)} \exp \left(2 \sum_{\substack{m,n=0 \\ (m,n) \neq (0,0)}}^{\infty} [B_{m,n}(\sigma) - B_{n,m}(-\sigma)] x^{2\alpha_{m,n}} \right) \quad (3.34)$$

Obviously $B_{m,0}(\sigma) = 0$ for $m > 0$. Few first $B_{m,n}$'s are

$$\begin{aligned} B_{0,1}(\sigma) &= \frac{4}{\kappa(\sigma)(1-\sigma)^2} \\ B_{1,1}(\sigma) &= -\frac{8}{(1+\sigma)^2} \\ B_{0,2}(\sigma) &= \frac{8}{\kappa^2(\sigma)(1-\sigma)^4} \end{aligned} \quad (3.35)$$

The polymer one-loop scaling functions of sect.2 correspond to the special case $\sigma = 1/3$.

4. Relation to sin-Gordon

The sin-Gordon model is a theory of 2D scalar field $\varphi(x)$, $x^\mu = (x^0, x^1)$, with the action

$$A_{sG} = \frac{1}{2} \int (\partial_\mu \varphi)^2 d^2x - \mu \int : \cos \beta \varphi : d^2x \quad (4.1)$$

where β is a real dimensionless parameter ($\beta^2 \leq 8\pi$). Here $: \dots :$ denotes the normal ordering with respect to massless free fields. To be precise we choose such an infrared cutoff that at $\mu = 0$ the two-point correlation function $\langle \varphi(x) \varphi(0) \rangle$ is

$$\langle \varphi(x) \varphi(0) \rangle_{\mu=0} = -\frac{1}{2\pi} \log |x| \quad (4.2)$$

In this scheme the (real) coupling constant μ is dimensional $\mu \sim [\text{mass}]^{2-2\Delta}$ with

$$\Delta = \frac{\beta^2}{8\pi} \quad (4.3)$$

Sometimes it is convenient to use another (positive) parameter

$$p = \frac{\beta^2}{8\pi - \beta^2} \quad (4.4)$$

instead of β .

The action (4.1) is obviously symmetric under the global shifts of the field φ

$$\varphi(x) \rightarrow \varphi(x) + \frac{2\pi n}{\beta} \quad (4.5)$$

where n is an x -independent integer. Therefore the states $|\Psi\rangle$ of the model can be classified in their behavior under (4.5). The irreducible states $|\Psi_\alpha\rangle$ are characterized by the index (quasimomentum) $-\beta/2 < \alpha \leq \beta/2$ such that

$$\Psi_\alpha\left(\varphi + \frac{2\pi}{\beta}\right) = e^{2i\pi\alpha/\beta} \Psi_\alpha(\varphi) \quad (4.6)$$

For each α there is an infinite dimensional space of states \mathcal{A}_α , the complete space being

$$\mathcal{A} = \bigoplus_{-\beta/2 < \alpha \leq \beta/2} \mathcal{A}_\alpha \quad (4.7)$$

In the infinite volume system the transitions (4.5) are suppressed to zero so that all the spaces \mathcal{A}_α are completely degenerate in α . However in a finite geometry (we imply that the finite geometry settlement respects the symmetry (4.5)) these transitions are allowed and α becomes an important parameter of the theory. Let us call \mathcal{A}_α the α -sector and the corresponding ground state $|\Psi_\alpha^{(0)}\rangle$ the α -ground state. The α -ground state energy E_α is a periodic function of the quasimomentum $\alpha \sim \alpha + \beta\mathbb{Z}$. For symmetry reasons we expect E_α to have a minimum at $\alpha = 0 \pmod{\beta\mathbb{Z}}$, i.e. $|\Psi_0^{(0)}\rangle$ is the true ground state.

Before considering the finite geometry effects remind briefly the well known structure of the infinite volume space \mathcal{A}_α which is essentially independent on α . \mathcal{A}_α contains the infinite volume ground state (the α -vacuum) together with the excitations which are (massive at $\mu \neq 0$ in (4.1)) relativistic particles subject a to factorized scattering. The spectrum of particles always contains the soliton-antisoliton doublet (s, \bar{s}) . Its mass M is related to the coupling μ in eq.(4.1) as follows [8]

$$\pi\mu = \frac{\Gamma\left(\frac{p}{p+1}\right)}{\Gamma\left(\frac{1}{p+1}\right)} \left[\frac{\sqrt{\pi}M\Gamma\left(\frac{p+1}{2}\right)}{2\Gamma\left(\frac{p}{2}\right)} \right]^{2/(p+1)} \quad (4.8)$$

Quasiclassically one can think about s and \bar{s} as of the kink configurations of field $\varphi(x)$ such that (currently we imply x^1 to be the spatial coordinate)

$$\varphi(x^1 \rightarrow \infty) - \varphi(x^1 \rightarrow -\infty) = \pm 2\pi/\beta \quad (4.9)$$

for the soliton and antisoliton respectively. At $p < 1$ the spectrum includes also the $s - \bar{s}$ bound states B_n , $n = 1, 2, \dots < 1/p$ of masses

$$m_n = 2M \sin \frac{\pi p n}{2} \quad (4.10)$$

The factorized scattering amplitudes of all these particles can be found e.g. in [12]. Note that all the observables are independent on α . The (specific) ground state energy [7,8]

$$\mathcal{E} = -\frac{M^2}{4} \tan \frac{\pi p}{2} \quad (4.11)$$

is also α independent.

Let us now put the sin-Gordon model on a finite space circle of circumference R and impose the periodic boundary conditions, i.e., $\varphi(x^0, x^1 + R) = \varphi(x^0, x^1)$. In the euclidean version of (4.1) this corresponds to the geometry of infinite (or very long $L \rightarrow \infty$ in the “time” direction x^0) flat cylinder. We denote $E_\alpha(R)$ the corresponding α -ground state energy. In the perturbed conformal field theory (CFT) picture this observable would correspond to the specific (per unit length of the cylinder) free energy with the scalar operator $\exp(i\alpha\varphi)$ “flowing along the cylinder”. The UV conformal dimension of this operator is $\Delta_\alpha = \alpha^2/8\pi$. Therefore we expect at $R \rightarrow 0$ [13]

$$E_\alpha(R) \sim -\frac{\pi c_\alpha}{6R} = -\frac{\pi}{6R} \left(1 - \frac{3\alpha^2}{\pi}\right) \quad (4.12)$$

In general

$$E_\alpha(R) = M F_{sG}(\alpha, p | MR) \quad (4.13)$$

where the soliton mass M is used as an overall scale to isolate the scale independent function F_{sG} and the dependence on the sin-Gordon parameter (4.4) is explicitly indicated. From (4.12) we have at $t \rightarrow 0$

$$F_{sG}(\alpha, p | t) \sim -\frac{\pi}{6t} \left(1 - \frac{3\alpha^2}{\pi}\right), \quad t \rightarrow 0 \quad (4.14)$$

independently on p . In the opposite limit $t \rightarrow \infty$ we expect from (4.11)

$$F_{sG}(\alpha, p | t) = -\frac{t}{4} \tan \frac{\pi p}{2} + \text{corrections}, \quad t \rightarrow \infty \quad (4.15)$$

independently on α . The leading correction in (4.15) comes (at $p > 1/3$) from the virtual soliton or antisoliton trajectory winding once around the cylinder. In view of (4.9) it is plain that in the α -sector these trajectories are weighted by the factors of $\exp(\pm 2i\pi\alpha/\beta)$ respectively. Therefore, summing up the s and \bar{s} contributions

$$F_{sG}(\alpha, p | t) + \frac{t}{4} \tan \frac{\pi p}{2} = -\frac{2}{\pi} \cos \frac{2\pi\alpha}{\beta} K_1(t) + (\text{multiparticle or bound state contributions}) \quad (4.16)$$

Considering at the microscopic level one can argue [14] that the described above cylinder sin-Gordon settlement is in the same universality class as the cylinder polymer counting problem of sect.2 provided the non-winding polymer weight n is related to the sin-Gordon parameter (4.4) by eq.(1.5). Also, comparing (4.16) with what one expects at

large t for the polymer counting scaling function $F(m, n|t)$ we can relate the sin-Gordon quasimomentum $2\pi\alpha/\beta$ to the winding polymer weight m

$$m = 2 \cos \frac{2\pi\alpha}{\beta} \quad (4.17)$$

In addition it is clear that the soliton mass M has to be identified with the inverse correlation length of the polymer problem. Altogether it seems not misleading to think of the polymer loops as of the virtual soliton trajectories (summed over the “orientations” s and \bar{s}). Note, that the sin-Gordon–polymer relation (1.5) implies that $p > 1$ where the sin-Gordon spectrum contains no extra particles but s and \bar{s} . It looks quite natural that the self-avoiding polymers form no bound states.

We conclude that if m and n are related to p and α as in eqs.(1.5) and (4.17)

$$F(m, n|t) = F_{sG}(\alpha, p|t) \quad (4.18)$$

This correspondence turns rather useful. It permits often to reinterpret the sin-Gordon results in the polymer terms. E.g., eq.(1.7) is simply read off from the ground state energy (4.11). Vice versa (4.18) implies that at $m = n = 0$, i.e. $p = 2$ and $2\alpha/\beta = 1/2$ the sin-Gordon α -ground state energy $F_{sG}(\alpha, p|t)$ vanishes for all t . This is indeed the case due to the $N = 2$ supersymmetry of the sin-Gordon model at this point [15]. One then can expand $F_{sG}(\alpha, p|t)$ around this point to obtain the scaling functions $F_{l_1, l_2}(t)$ in eq.(2.9).

The standard tools of studying the sin-Gordon model become applicable to the polymer problem of sect.2. In particular, in the next section we use the perturbation theory of (4.1) in μ to evaluate the small t behavior of $F(m, n|t)$. The perturbative series for $E_\alpha(R)$ is constructed in the usual way. In fact it is the expansion

$$E_\alpha(R) = -\frac{\pi}{6R} \sum_{n=0}^{\infty} c_{2n}(\alpha) \eta^{2n} \quad (4.19)$$

in the dimensionless variable

$$\eta = -2\pi\mu \left(\frac{R}{2\pi} \right)^{2/(p+1)} = -2\pi r_p \left(\frac{t}{2\pi} \right)^{2/(p+1)} \quad (4.20)$$

where

$$r_p = \mu M^{-2/(p+1)} \quad (4.21)$$

is the “ μ - M ratio” defined by eq.(4.8). The first coefficient in (4.19)

$$c_0(\alpha) = 1 - \frac{3\alpha^2}{\pi} \quad (4.22)$$

is the effective central charge corresponding to operator $\exp(i\alpha\varphi)$ while the subsequent ones are given by perturbative integrals. Introducing the complex coordinate $z = x^1 + ix^0$

on the cylinder $z \sim z + R\mathbb{Z}$ and substituting $u = \exp(2\pi iz/R)$ we have explicitly

$$c_{2n}(\alpha) = \frac{12}{(2\pi)^{2n-1}(n!)^2} \times \int_{(u_1=1)} \left[\frac{\prod_{i>j}^n |u_i - u_j|^{4\Delta} |v_i - v_j|^{4\Delta}}{\prod_{i,j}^n |u_i - v_j|^{4\Delta}} \prod_{i=1}^n |u_i|^{2\Delta(1-2\alpha/\beta)} |v_i|^{2\Delta(1+2\alpha/\beta)} - \text{subtractions} \right] \frac{d^2 u_i}{|u_i|^2} \frac{d^2 v_i}{|v_i|^2} \quad (4.23)$$

where the disconnected parts are subtracted to build the connected $2n$ -point cylinder correlation function. The integrand is invariant under the homogeneous shifts along the cylinder $u_i \rightarrow \Lambda u_i$, $v_i \rightarrow \Lambda v_i$ so that we can fix one of the points (u_1 in eq.(4.23)). Note that the perturbative coefficients $c_{2n}(\alpha)$ are not periodic in the quasimomentum α . In fact expansion (4.19) determines the α -ground state energy only if $|\alpha| < \beta/2$. For $|\alpha| > \beta/2$ and $\alpha = \alpha_0 \pmod{\beta}$, $|\alpha_0| < \beta/2$ it corresponds instead to an excited state in the α_0 sector. The discontinuities occur at $\alpha = n\beta/2$ with n integer, where the unperturbed space of states contains two (connected perturbatively) degenerate states $\exp(\pm in\beta\varphi/2)$ and expansion (4.19) has to be modified.

The series (4.19) admits another interpretation if $2\alpha/\beta = \pm(1/p - k)$ with k a non-negative integer [16] (note that in this series only $k = 0$ and 1 would correspond to α -ground states). Take e.g., $2\alpha/\beta = 1/p - k$ and consider first the integration over v_i , $i = 1, 2, \dots, n$ in eq.(4.23). Following [17] we have

$$\frac{1}{n!} \int \frac{\prod_{i>j}^n |u_i - u_j|^{4p/(p+1)} |v_i - v_j|^{4p/(p+1)}}{\prod_{i,j}^n |u_i - v_j|^{4p/(p+1)}} \prod_{i=1}^n |u_i|^{2(kp-2)/(p+1)} |v_i|^{-2kp/(p+1)} d^2 v_i = \quad (4.24)$$

$$\left(\frac{2\pi}{Q_p} \right)^n \langle \Phi_{1,k+1}(0) \Phi_{1,3}(u_1) \Phi_{1,3}(u_2) \dots \Phi_{1,3}(u_n) \Phi_{1,k+1}(\infty) \rangle_{\mathcal{M}_p} \prod_{i=1}^n |u_i|^{-4/(p+1)}$$

where $\Phi_{1,k}$, $k = 1, 2, \dots$ denote the (thermal series) primary fields in the minimal CFT \mathcal{M}_p and $\langle \dots \rangle_{\mathcal{M}_p}$ is quite the correlation function in this model. Corresponding to the usual in CFT normalization of fields $\langle \Phi_{1,k}(x) \Phi_{1,k}(0) \rangle_{\mathcal{M}_p} = |x|^{-4\Delta_k}$ (here $4\Delta_k = k(kp-2)/(p+1)$) we have to take

$$Q_p = \frac{2(2p-1)(p-1)}{(p+1)^2} \left[\frac{\Gamma^3\left(\frac{p}{p+1}\right) \Gamma\left(\frac{1-2p}{p+1}\right)}{\Gamma^3\left(\frac{1}{p+1}\right) \Gamma\left(\frac{3p}{p+1}\right)} \right]^{1/2} \quad (4.25)$$

Thus denoting

$$\lambda = -\frac{2\pi\mu^2}{Q_p} \quad (4.26)$$

one can interpret the series (4.19) as the perturbative expansion in the Φ_{13} -perturbed \mathcal{M}_p , i.e., in the RSG model

$$A_\lambda = A_{\mathcal{M}_p} + \lambda \int \Phi_{13}(x) d^2x \quad (4.27)$$

where $A_{\mathcal{M}_p}$ is the formal action of the CFT \mathcal{M}_p . Under this interpretation the quantity (4.19) at $2\alpha/\beta = 1/p - k$ becomes the perturbed finite size energy of the primary state $\Phi_{1,k+1}$ in the model (4.27). In particular $k = 0$ and $k = 1$ correspond to the perturbed states $\Phi_{11} = \mathbb{I}$ and Φ_{12} respectively. If $2\alpha/\beta = -1/p + k$ we can similarly consider first the integration over u_i in (4.23) and then treat the remaining integral over v_i as a perturbative one in the model (4.27).

Let us denote $F_{(k)}(p|t)$ the scaling function corresponding to the finite size energy of the RSG state $\Phi_{1,k+1}$. If the sin-Gordon coupling μ is related to λ as in eq.(4.26) we have

$$F_{(k)}(p|t) = F_{sG}(\pm(1/p - k), p|t) \quad (4.28)$$

Comparing with (4.18) we see that the polymer functions (2.7) and (2.8) are (provided relation (1.5) between p and n holds)

$$\begin{aligned} F_+(n|t) &= F_{(0)}(p|t) \\ F_-(n|t) &= F_{(1)}(p|t) \end{aligned} \quad (4.29)$$

i.e. correspond to the RSG states \mathbb{I} and Φ_{12} respectively.

5. Perturbation theory

Here we present few explicit calculations about the sin-Gordon perturbative series (4.19). For the later comparisons it is convenient to use the scaling parameter

$$\xi = (2\pi)^{-2(p-1)/(p+1)} \eta^2 / r_p^2 \quad (5.1)$$

instead of η in eq.(4.19) (here r_p is the μ - M ratio (4.8), (4.12)). At $p = 2$ this notation conforms that of eq.(2.17). From (4.19) we have

$$F_{sG}(\alpha, p|t) = -\frac{\pi}{6t} \sum_{n=0}^{\infty} c_{2n}(\alpha) \left[(2\pi)^{(p-1)/(p+1)} r_p \right]^{2n} \xi^n \quad (5.2)$$

At $n = 1$ the integral (4.23) is carried out explicitly

$$c_2(\alpha) = 6 \frac{\gamma\left(\frac{p}{p+1} \left(1 - \frac{2\alpha}{\beta}\right)\right) \gamma\left(\frac{p}{p+1} \left(1 + \frac{2\alpha}{\beta}\right)\right)}{\gamma\left(\frac{2p}{p+1}\right)} \quad (5.3)$$

where the notation $\gamma(x) = \Gamma(x)/\Gamma(1-x)$ is used.

In the polymer language eq.(5.2) together with (4.22) and (5.3) predicts the leading $t \rightarrow 0$ behavior and the next-to-leading correction to the polymer scaling function $F(m, n|t)$. In particular, expanding around $p = 2$ and $2\alpha/\beta = 1/2$ we find

$$tF(m, n|t) = m \left(-\frac{1}{3} + \frac{3}{4\gamma^2(1/3)}\xi + \dots \right) + n \left(\frac{1}{18} - \frac{3}{4\gamma^2(1/3)}\xi + \dots \right) + \dots \quad (5.4)$$

in agreement with eqs.(2.19) and (2.30).

At $2\alpha/\beta = 1/p$ (this value corresponds to $m = n$ in the polymer language) we can go two steps further thanks to the RSG interpretation described in sect.4. Namely

$$\begin{aligned} c_4 \left(\frac{2\alpha}{\beta} = \frac{1}{p} \right) &= \frac{3}{\pi Q_p^2} \int \langle \Phi_{13}(1) \Phi_{13}(u) \rangle_{\mathcal{M}_p} |u|^{-4/(p+1)} d^2 u \\ &= \frac{3(p+1)^4 \gamma^2 \left(\frac{p-1}{p+1} \right) \gamma \left(\frac{3p}{p+1} \right)}{4(p-1)^2 (2p-1)^2 \gamma \left(\frac{2p-2}{p+1} \right) \gamma^3 \left(\frac{p}{p+1} \right)} \end{aligned} \quad (5.5)$$

and

$$\begin{aligned} c_6 \left(\frac{2\alpha}{\beta} = \frac{1}{p} \right) &= \frac{1}{2\pi^2 Q_p^3} \int \langle \Phi_{13}(1) \Phi_{13}(u_1) \Phi_{13}(u_2) \rangle_{\mathcal{M}_p} |u_1 u_2|^{-4/(p+1)} d^2 u_1 d^2 u_2 \\ &= - \frac{18(p+1)^4 \gamma \left(\frac{p-1}{p+1} \right) \gamma^2 \left(\frac{3p}{p+1} \right) \gamma \left(\frac{2p-2}{p+1} \right)}{(p-1)^2 (2p-1)^2 \gamma \left(\frac{5p-1}{p+1} \right) \gamma^6 \left(\frac{p}{p+1} \right)} \end{aligned} \quad (5.6)$$

At $p = 2 + 2n/\pi + O(n^2)$ this gives for the RSG scaling function $F_{(0)}(p|t)$ (see eq.(4.28))

$$tF_{(0)}(p|t) = n \left(-\frac{5}{18} - \frac{3\gamma^2(1/3)}{64}\xi^2 + \frac{9}{256}\xi^3 + \dots \right) + O(n^2) \quad (5.7)$$

in complete accordance with the expansion (2.31).

Another special case of the oppositely weighted winding and non-winding polymer loops $m = -n$ is related by eq.(4.17) to the sin-Gordon quasimomentum $2\alpha/\beta = \pm(1/p-1)$. This point again admits the RSG interpretation corresponding now to the finite-size ground state in the Φ_{12} sector of the Φ_{13} -perturbed \mathcal{M}_p . Therefore

$$c_4 \left(\frac{2\alpha}{\beta} = \frac{1}{p} - 1 \right) = \frac{3}{\pi Q_p^2} \int \ll \Phi_{12}(0) \Phi_{13}(1) \Phi_{13}(u) \Phi_{12}(\infty) \gg_{\mathcal{M}_p} \frac{d^2 u}{|u|^2} \quad (5.8)$$

where $\ll \dots \gg_{\mathcal{M}_p}$ denotes the connected cylinder correlation function in \mathcal{M}_p . Explicitly

$$\ll \Phi_{12}(0) \Phi_{13}(1) \Phi_{13}(u) \Phi_{12}(\infty) \gg_{\mathcal{M}_p} = a_1 (f_1(u) f_1(\bar{u}) - 1) + a_2 f_2(u) f_2(\bar{u}) \quad (5.9)$$

where

$$a_1 = \frac{\gamma\left(\frac{p}{p+1}\right)\gamma\left(\frac{2p-1}{p+1}\right)}{\gamma\left(\frac{2p}{p+1}\right)\gamma\left(\frac{p-1}{p+1}\right)}; \quad a_2 = \frac{\gamma\left(\frac{p}{p+1}\right)\gamma\left(\frac{1-2p}{p+1}\right)}{\gamma\left(\frac{2-2p}{p+1}\right)\gamma\left(\frac{p-1}{p+1}\right)} \quad (5.10)$$

and $f_{1,2}$ are in terms of the hypergeometric functions

$$\begin{aligned} f_1(u) &= (1-u)^{(2-2p)/(p+1)} F\left(\frac{2-2p}{p+1}, \frac{1}{p+1}, \frac{2-p}{p+1}, u\right) \\ f_2(u) &= u^{(2p-1)/(p+1)} (1-u)^{(2-2p)/(p+1)} F\left(\frac{2p}{p+1}, \frac{1}{p+1}, \frac{3p}{p+1}, u\right) \end{aligned} \quad (5.11)$$

The integral (5.8) again can be carried out

$$\begin{aligned} c_4\left(\frac{2\alpha}{\beta} = \frac{1}{p} - 1\right) &= \frac{3\gamma^2\left(\frac{2p-1}{p+1}\right)}{2\gamma^2\left(\frac{2p}{p+1}\right)\gamma^2\left(\frac{p}{p+1}\right)} \left[\psi\left(\frac{2-p}{p+1}\right) + \right. \\ &\quad \left. \psi\left(\frac{2p-1}{p+1}\right) - \psi\left(\frac{p}{p+1}\right) - \psi\left(\frac{1}{p+1}\right) - \psi\left(\frac{3p-1}{2(p+1)}\right) - \psi\left(\frac{3-p}{2(p+1)}\right) - 2\psi\left(\frac{1}{2}\right) \right] \end{aligned} \quad (5.12)$$

At $p \rightarrow 2$ we have therefore

$$tF_{(1)}(p|t) = n \left(\frac{7}{18} - \frac{3}{2\gamma^2(1/3)}\xi - \frac{27}{32\gamma^4(1/3)}\xi^2 + \dots \right) + O(n^2) \quad (5.13)$$

to be compared with the first terms of eq.(2.32).

6. TBA considerations

In the previous two sections the sin-Gordon scaling function $F_{sG}(\alpha, p|t)$ has been studied in the UV perturbation theory. In principle this approach provides us with a systematic expansion of $F_{sG}(\alpha, p|t)$ in powers of $t^{2/(p+1)}$. At the same time the integrability of the sin-Gordon model allows the same observable F_{sG} to be considered in the TBA framework. In TBA one does not concern directly the field theory action starting instead from the (exactly known in the sin-Gordon case) relativistic factorized scattering theory. After some manipulations the cylinder scaling function is related to a system of non-linear integral equations (the TBA system), its form depending on the factorized scattering amplitudes.

In the case of the sin-Gordon scattering theory the TBA system (at $p > 1$) is borrowed essentially from the construction by Takahashi and Suzuki [18] for the anisotropic Heisenberg chain. The structure depends drastically on the arithmetic nature of the sin-Gordon parameter p . In general it is an infinite system of coupled non-linear integral equations for

infinite number of unknown functions. At p rational it can be simplified and reduced to a finite system (see [18] for the details). This is why in the TBA studies of the sin-Gordon model one tends to choose p rational. It should be mentioned however that this “fractal” dependence on p is a problem of the TBA technique itself. The sin-Gordon physics is quite continuous in the coupling constant.

For the time being we are interested in the point $p = 2$ and its vicinity. At $p = 2$ the TBA system contains only two unknown functions $e_1(\theta)$ and $e_2(\theta)$ and reads

$$\begin{aligned} t \cosh \theta &= e_1 + s * \log(1 + 2 \cos(4\pi\alpha/\beta) e^{-e_2} + e^{-2e_2}) \\ 0 &= e_2 + s * \log(1 + e^{-e_1}) \end{aligned} \quad (6.1)$$

where $*$ denotes the convolution in θ

$$s * g = \int_{-\infty}^{\infty} s(\theta - \theta') g(\theta') d\theta' \quad (6.2)$$

with the kernel

$$s(\theta) = \int \frac{e^{i\omega\theta}}{2 \cosh \frac{\pi\omega}{2}} \frac{d\omega}{2\pi} = \frac{1}{2\pi \cosh \theta} \quad (6.3)$$

The sin-Gordon scaling function appears as

$$F_{sG}(\alpha, 2|t) = -\frac{1}{2\pi} \int \cosh \theta \log(1 + e^{-e_1(\theta)}) d\theta \quad (6.4)$$

It happens that at $m = 2 \cos(2\pi\alpha/\beta)$ small, $e_1 \sim -\log m$ and $e_2 \sim m$. Denoting

$$\begin{aligned} e^{-e_1(\theta)} &= m e^{-\varepsilon(\theta)} \\ e^{-e_2(\theta)} &= 1 + m \eta(\theta) \end{aligned} \quad (6.5)$$

we find that at $m \rightarrow 0$

$$F_{sG}(\alpha, 2|t) = -m \int \cosh \theta e^{-\varepsilon(\theta)} \frac{d\theta}{2\pi} + O(m^2) \quad (6.6)$$

while $\varepsilon(\theta)$ and $\eta(\theta)$ solve the following system

$$\begin{aligned} t \cosh \theta &= \varepsilon + s * \log(1 + \eta^2) \\ \eta &= s * e^{-\varepsilon} \end{aligned} \quad (6.7)$$

which is quite the one quoted in sect.3 eq.(3.21) (with $\lambda = 1/4\pi$). Note, that eqs.(6.7) imply the following functional system for $R_+(\theta) = \exp(-\varepsilon(\theta))$ and $\eta(\theta)$

$$\begin{aligned} R_+(\theta + i\pi/2) R_+(\theta - i\pi/2) &= 1 + \eta^2(\theta) \\ \eta(\theta + i\pi/2) + \eta(\theta - i\pi/2) &= R_+(\theta) \end{aligned} \quad (6.8)$$

Moreover, it can be shown that under appropriate analytic and asymptotic restrictions on $R_+(\theta)$ and $\eta(\theta)$ the functional and integral systems (6.8) and (6.7) are equivalent.

To arrive at eq.(3.23) one needs to shift slightly from the point $p = 2$. For the reasons mentioned above and following ref.[1] we choose the series of rational values

$$p = 2 + 1/N, \quad N = 2, 3, 4, \dots \quad (6.9)$$

Any analytic at $p = 2$ function of p is unambiguously recovered from the series of its values at (6.9).

The Takahashi-Suzuki TBA system at $p = 2 + 1/N$ is quoted in [1]. It includes $N + 2$ functions $\varepsilon_a(\theta)$; $a = 0, 1, \dots, N + 1$ and has the form

$$\begin{aligned} t \cosh \theta &= \varepsilon_0 + s * L_1 \\ 0 &= \varepsilon_1 + s * L_0 + s_N * L_2 + d_N * L_1 \\ 0 &= \varepsilon_2 + s_N * [L_3 - L_1] \\ 3 \leq a \leq N, \quad 0 &= \varepsilon_a + s_N * [L_{a+1} + L_{a-1}] \\ 0 &= \varepsilon_{N+1} + s_N * L_N \end{aligned} \quad (6.10)$$

The kernel $s(\theta)$ is given by eq.(6.3),

$$\begin{aligned} s_N(\theta) &= \int \frac{e^{i\omega\theta}}{2 \cosh \frac{\pi\omega}{2N}} \frac{d\omega}{2\pi} \\ d_N(\theta) &= \int \frac{e^{i\omega\theta} \cosh \frac{\pi(N-1)\omega}{2N}}{2 \cosh \frac{\pi\omega}{2} \cosh \frac{\pi\omega}{2N}} \frac{d\omega}{2\pi} \end{aligned} \quad (6.11)$$

while the functions $L_a(\theta)$; $a = 0, 1, 2, \dots, N + 1$ in eq.(6.10) are read as follows

$$\begin{aligned} L_a(\theta) &= \log \left(1 + e^{-\varepsilon_a(\theta)} \right), \quad 0 \leq a \leq N \\ L_{N+1}(\theta) &= \log \left[\left(1 + qe^{-\varepsilon_{N+1}(\theta)} \right) \left(1 + q^{-1}e^{-\varepsilon_{N+1}(\theta)} \right) \right] \end{aligned} \quad (6.12)$$

Here

$$q = \exp \left(\frac{2i\pi\alpha}{\beta} (2N + 1) \right) \quad (6.13)$$

introduces the sin-Gordon quasimomentum α . The cylinder scaling function is determined by $L_0(\theta)$

$$F_{sG}(\alpha, 2 + 1/N | t) = -\frac{1}{2\pi} \int \cosh \theta L_0(\theta) d\theta - \frac{t}{4} \tan \frac{\pi}{2N} \quad (6.14)$$

For our purposes it is more convenient to consider the functional system for $Y_a(\theta) = \exp(-\varepsilon_a(\theta))$, $a = 0, 1, 2, \dots, N + 1$, which follows from (6.10)

$$Y_0 \left(\theta + \frac{i\pi}{2} \right) Y_0 \left(\theta - \frac{i\pi}{2} \right) = 1 + Y_1(\theta) ;$$

$$\begin{aligned}
& \frac{Y_1 \left(\theta + \frac{i\pi(N+1)}{2N} \right) Y_1 \left(\theta - \frac{i\pi(N+1)}{2N} \right)}{\left(1 + Y_1^{-1} \left(\theta + \frac{i\pi(N-1)}{2N} \right) \right) \left(1 + Y_1^{-1} \left(\theta - \frac{i\pi(N-1)}{2N} \right) \right)} = \\
& \left(1 + Y_0 \left(\theta + \frac{i\pi}{2N} \right) \right) \left(1 + Y_0 \left(\theta - \frac{i\pi}{2N} \right) \right) \left(1 + Y_2 \left(\theta + \frac{i\pi}{2} \right) \right) \left(1 + Y_2 \left(\theta - \frac{i\pi}{2} \right) \right) ; \\
& Y_2 \left(\theta + \frac{i\pi}{2N} \right) Y_2 \left(\theta - \frac{i\pi}{2N} \right) = \frac{1 + Y_3(\theta)}{1 + Y_1(\theta)} ; \\
& Y_a \left(\theta + \frac{i\pi}{2N} \right) Y_a \left(\theta - \frac{i\pi}{2N} \right) = (1 + Y_{a+1}(\theta)) (1 + Y_{a-1}(\theta)) , \quad 2 \leq a \leq N-1 ; \\
& Y_N \left(\theta + \frac{i\pi}{2N} \right) Y_N \left(\theta - \frac{i\pi}{2N} \right) = (1 + Y_{N-1}(\theta)) (1 + qY_{N+1}(\theta)) (1 + q^{-1}Y_{N+1}(\theta)) ; \\
& Y_{N+1} \left(\theta + \frac{i\pi}{2N} \right) Y_{N+1} \left(\theta - \frac{i\pi}{2N} \right) = 1 + Y_N(\theta)
\end{aligned} \tag{6.15}$$

It is easy to verify that this system is “solved” in terms of a single function $G(\theta)$ (defined up to an overall multiplying constant) as

$$\begin{aligned}
Y_0(\theta) &= \frac{(q^2 G_{2N+1}(\theta) - q^{-2} G_{-2N-1}(\theta)) (G_1(\theta) - G_{-1}(\theta))}{(q G_{2N+1}(\theta) - q^{-1} G_1(\theta)) (q G_{-1}(\theta) - q^{-1} G_{-2N-1}(\theta))} \\
1 + Y_1(\theta) &= \frac{(G_{N+1}(\theta) - G_{N-1}(\theta)) (G_{-N+1}(\theta) - G_{-N-1}(\theta))}{(q G_{N-1}(\theta) - q^{-1} G_{-N-1}(\theta)) (q G_{N+1}(\theta) - q^{-1} G_{-N+1}(\theta))} \\
Y_{N+1}(\theta) &= -\frac{q G_1(\theta) - q^{-1} G_{-1}(\theta)}{G_1(\theta) - G_{-1}(\theta)}
\end{aligned} \tag{6.16}$$

and for $2 \leq a \leq N$

$$1 + Y_a(\theta) = \frac{(q G_{N-a}(\theta) - q^{-1} G_{-N+a-2}(\theta)) (q G_{N-a+2}(\theta) - q^{-1} G_{-N+a}(\theta))}{(G_{N-a+2}(\theta) - G_{N-a}(\theta)) (G_{-N+a}(\theta) - G_{-N+a-2}(\theta))} \tag{6.17}$$

provided $G(\theta)$ satisfies

$$G_{6N+2}(\theta) = q^{-6} G(\theta) \tag{6.18}$$

In eqs.(6.16–18) we use the abbreviation

$$G_k(\theta) = G \left(\theta + \frac{i\pi k}{2N} \right) \tag{6.19}$$

Note that due to (6.18) all the functions $Y_a(\theta)$ are $i\pi(3 + 1/N)$ -periodic in θ .

Turn at the limit $N \rightarrow \infty$ and $2\alpha/\beta = 1/2 - \epsilon/2$ with $\epsilon \rightarrow 0$. Following eqs.(1.5) and (4.17)

$$\begin{aligned}
m &= \pi\epsilon + O(\epsilon^2) \\
n &= \frac{\pi}{2N} + O \left(\frac{1}{N^2} \right)
\end{aligned} \tag{6.20}$$

In this notation $q = i(-)^N \exp(-i\pi\epsilon(N+1/2))$. It is convenient to introduce new function

$$h(\theta) = G(\theta) \exp\left(-\frac{3N(2N+1)\epsilon}{2N+1}\theta\right) \quad (6.21)$$

which enjoys the m, n -independent behavior under the period shift (6.18)

$$h(\theta + i\pi(3 + 1/N)) = -h(\theta) \quad (6.22)$$

The function $Y_0(\theta)$, which is most important for us, now becomes (again $h_k(\theta) = h(\theta + i\pi k/2N)$)

$$Y_0(\theta) = \frac{(e^{i\delta/3}h_{2N+1}(\theta) - e^{-i\delta/3}h_{-2N-1}(\theta))(e^{i\delta}h_1(\theta) - e^{-i\delta}h_{-1}(\theta))}{(e^{i\delta/3}h_{2N+1}(\theta) + e^{i\delta}h_1(\theta))(e^{-i\delta}h_{-1}(\theta) + e^{-i\delta/3}h_{-2N-1}(\theta))} \quad (6.23)$$

with $\delta = \pi\epsilon(N+1/2)/(N+1/3)$. At $N = \infty$ and $\epsilon = 0$ this quantity vanishes as one could expect, while $h(\theta)$ turns to some limiting function $g(\theta)$ such that

$$g(\theta + 3i\pi) = -g(\theta) \quad (6.24)$$

Moreover, the first order terms in m and n

$$Y_0(\theta) = mR_+(\theta) - nR_-(\theta) + \text{higher order terms} \quad (6.25)$$

are determined by $g(\theta)$

$$R_+(\theta) = \frac{2i(g(\theta + i\pi) - g(\theta - i\pi))g(\theta)}{(g(\theta + i\pi) + g(\theta))(g(\theta) + g(\theta - i\pi))} \quad (6.26a)$$

$$R_-(\theta) = -\frac{2i(g(\theta + i\pi) - g(\theta - i\pi))g'(\theta)}{(g(\theta + i\pi) + g(\theta))(g(\theta) + g(\theta - i\pi))} \quad (6.26b)$$

Denoting also

$$\eta(\theta) = -i\frac{g(\theta + i\pi/2) - g(\theta - i\pi/2)}{g(\theta + i\pi/2) + g(\theta - i\pi/2)} \quad (6.27)$$

one finds that (6.26a) and (6.27) satisfy the functional system (6.8). Therefore we have to identify these functions with the solutions to the TBA system (6.7).

From (6.27) one can find $g(\theta)$ in terms of $\eta(\theta)$

$$\log g(\theta) = -\frac{1}{\pi} \int \tanh(\theta - \theta') \arctan \eta(\theta') d\theta' + \text{const} \quad (6.28)$$

Relation (3.23) follows.

The TBA system (6.7) together with (3.23) has been used to compute numerically

$$F_n(t) = -\frac{1}{2\pi} \int \cosh \theta R_+(\theta) d\theta \quad (6.29)$$

and

$$F_c(t) + \frac{t}{4} = \frac{1}{2\pi} \int \cosh \theta R_-(\theta) d\theta \quad (6.30)$$

for the plots of figs.5–7.

7. Concluding remarks

Expansions (2.19) and (2.28) imply that the scaling functions $tF_n(t)$ and $tF_c(t)$ are still real (at least at $|t|$ small enough) after the analytic continuation

$$t \rightarrow e^{3i\pi/4}t \quad (7.1)$$

(or $\xi \rightarrow -\xi$ in eq.(2.1)). As it is argued in [9] such an analytic continuation would correspond to the change from the subcritical (dilute) polymer phase to the supercritical (dense) scaling polymers. The continued functions $\tilde{F}_n(t) = e^{3i\pi/4}F_n(e^{3i\pi/4}t)$ and $\tilde{F}_c(t) = e^{3i\pi/4}F_c(e^{3i\pi/4}t)$ are given (inside the convergence region) by the alternated series (2.19) and (2.28) (again $\xi = t^{4/3}$)

$$\begin{aligned} t\tilde{F}_n(t) &= -\frac{1}{3} + \frac{4}{3} \sum_{n=1}^{\infty} (-)^n n U_n \xi^n \\ t\tilde{F}_c(t) &= \frac{1}{18} - \frac{4}{3} \sum_{n=1}^{\infty} (-)^n n V_n \xi^n \end{aligned} \quad (7.2)$$

with the same U_n and V_n as in sect.2. Outside the convergence region the “dense polymer” scaling functions can be found as a particular solution to the continued equation (2.13)

$$\frac{1}{t} \frac{d}{dt} t \frac{d}{dt} \tilde{U} = -\frac{1}{2} \cosh 2\tilde{U} \quad (7.3)$$

fixed by the initial condition at $t \rightarrow 0$

$$\tilde{U}(t) = -\frac{1}{3} \log \frac{t}{8} - \frac{1}{2} \log \kappa + O(t^{4/3}) \quad (7.4)$$

Now

$$t\tilde{F}_n(t) = t \frac{d\tilde{U}}{dt} \quad (7.5)$$

while $\tilde{F}_c(t)$ is restored from the continuation of eq.(2.23)

$$t\tilde{F}_c(t) = \frac{1}{18} + \frac{1}{2} \int_0^t \sinh 2\tilde{U}(r) r dr \quad (7.6)$$

Contrary to the dilute polymer case where $U(t)$ is always regular at real $t > 0$, the continued function $\tilde{U}(t)$ has an infinite number of singularities on the positive real axis. These appear

as double poles of $\exp(-2\tilde{U}(t))$ or as simple poles in $\tilde{F}_n(t)$ and $\tilde{F}_c(t)$. In ref.[9] these poles were attributed to the level-crossing effect observed in the finite-size dense polymer system. Note however that $\exp(2\tilde{U})$ has double zeroes instead of the poles and, as it is readily figured out from eq.(7.3), remains always finite at $t > 0$. Therefore $t\tilde{F}_-(t) = t\tilde{F}_c(t) - t\tilde{F}_n(t)$ (the continued $tF_-(t)$ of eq.(2.32)) is positive and non-singular with an infinite sequence of critical points.

Anyhow the analytically continued Painlevé III transcendent $\tilde{U}(t)$ worth more attention. Could one invent something as effective as the Fredholm or TBA-like representations (3.10) and (3.21) for this case?

A closely related continuation problem arises in connection with the general sin-Gordon scaling function (4.13). As one concludes from the structure (4.19) the continuation $\eta^2 \rightarrow -\eta^2$ (or equivalently $t \rightarrow e^{i\pi(p+1)/4}t$) leads to a real expansion for the function

$$t\tilde{F}_{sG}(\alpha, p|t) = tF_{sG}\left(e^{i\pi(p+1)/4}t\right) \quad (7.7)$$

From the perturbative point of view this alternated series would correspond to a purely imaginary coupling constant μ in the sin-Gordon action (4.1). Being defined this way the imaginary coupled sin-Gordon model (ISG) is apparently non-unitary and the common field theory intuition fails to figure out the structure of its space of states, vacuum etc. However one could proceed formally (say perturbatively) arriving at some sensible conclusions. E.g. the renormalization group calculations in ISG with $p \gg 1$ formally work and show that at least at p large enough ISG is massless and interpolates between two sin-Gordon critical points with different values of p . Moreover the local arguments about the sin-Gordon integrability are independent on the nature of μ . Therefore the same local higher-spin integrals of motion are expected in the ISG as well. In ref.[9] a consistent factorized scattering theory for the massless ISG excitations was proposed and supported by the calculation of the ground state energy in an external field. There are some difficulties with the TBA treatment of this scattering theory and up to now it is not clear if it is complete or other (massive) particles are present in the ISG spectrum.

The interest to ISG is not purely academic. As it was suggested in [9] this field theory model is closely related to the dense phase of the 2D polymer problem (in its scaling limit). In particular the renormalization group behavior of the ISG effective central charge conforms qualitatively that observed in the finite-size dense polymer system [1,9].

The TBA approach allows the finite-size dilute polymer scaling function to be evaluated very carefully. Unfortunately the lack of correct TBA equations impedes the same for the dense polymer case. In this connection I'd like to mention a new potential approach to the finite-size problem in the sin-Gordon model developed by Destri and deVega (DdV) [19]. Contrary to TBA one does not deal with the physical scattering theory, thermal equilibrium of physical particles etc., but starts instead with some “constituent particles” and their “bare” scattering. After a kind of renormalization DdV arrive at a system of integral equations which are free of the “bare” parameters (including instead the “renormalized” physical spectra and amplitudes) and resemble strongly the usual TBA equations. It is not yet clear if this approach can be generalized to other integrable relativistic models or how the Takahashi-Suzuki TBA equations are related to the DdV ones. However the

DdV system is verified (numerically) to predict the same finite-size energy as the TBA approach does, being at the same time in many respects much more convenient (in particular the DdV system is continuous in p , as opposed to the Takahashi-Suzuki TBA). Moreover a slight modification of the DdV system looks quite suitable for the analytic continuation (7.7) of the sin-Gordon scaling function. This modified DdV system will be reported elsewhere.

Finally, the following remark seems in order. The form of potential (3.9) suggests to consider a generalization

$$2u(\theta) = \dots + \bar{t}_3 e^{-3\theta} + \bar{t}_1 e^{-\theta} + t_1 e^{\theta} + t_3 e^{3\theta} + t_5 e^{5\theta} + \dots + r(\theta) \quad (7.8)$$

It is readily verified that in this case (in the same notations as in eq.(3.8))

$$\begin{aligned} \frac{\partial W_1}{\partial t_3} - \frac{\partial^3 W_1}{\partial t_1^3} &= 0 \\ \frac{\partial W_3}{\partial t_3} - \frac{\partial^3 W_3}{\partial t_1^3} &= -24 \left(\frac{\partial W_1}{\partial t_1} \right)^3 \end{aligned} \quad (7.9)$$

A natural guess is that in general the function (3.10) satisfies

$$\frac{\partial U}{\partial t_3} = \frac{\partial^3 U}{\partial t_1^3} - 2 \left(\frac{\partial U}{\partial t_1} \right)^3 \quad (7.10)$$

i.e. the modified KdV equation. Considering derivatives in t_5 etc. we can speculate some higher integrable differential equations of the KdV hierarchy, U being related to the corresponding τ -function.

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Figure Captions

Fig.1. Topologically different configurations of two closed polymers.

Fig.2. Infinite honeycomb cylinder.

Fig.3. Winding and non-winding polymers on a cylinder.

Fig.4. Distinct cylinder configurations of two closed polymer loops.

Fig.5. Non-contractible one-loop scaling function $tF_n(t)$.

Fig.6. Contractible one-loop scaling function $tF_c(t) + t^2/4$ with the bulk term subtracted.

Fig.7. Scaling functions $tF_+(t) + t^2/4$ (solid curve) and $tF_-(t) + t^2/4$ (dashed one).

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